JOURNAL OF MULTIVARIATE ANALYSIS 1, 394–411 (1971)

Direct Limits of Measure Spaces

SERGE VASILACH

Departement de Mathématiques, Université Laval, Quebec 10, Canada

Communicated by M. M. Rao

The present paper is devoted to the study of the direct limits of direct systems of measure (resp. probability) spaces. If I is a right directed preordered set, $(E_\alpha)_{\alpha \in I}$ a family of sets indexed by I, $G = \bigcup_{\alpha \in I} E_\alpha \times \{\alpha\}$ the sum of the family (E_α), $\mathfrak{M}_\alpha$ a σ-algebra in E_α for each $\alpha \in I$ and $M = \bigcup_{\alpha \in I} \mathfrak{M}_\alpha \times \{\{\alpha\}\}$ is the sum of the family $(\mathfrak{M}_\alpha)$, then it is shown that M is a σ-algebra in G. If $E = \varinjlim E_\alpha$ is the direct limit of the family (E_α), if $\hat{E} = \varinjlim \mathfrak{P}(E_\alpha)$ the direct limit of the family of power sets $(\mathfrak{P}(E_\alpha))$, if $\mathfrak{M} = \lim \mathfrak{M}_\alpha$ is the direct limit of the family $(\mathfrak{M}_\alpha)$, if $(E_\alpha, \mathfrak{M}_\alpha)$ is a direct system of measurable spaces, then $(E, \mathfrak{M}) = (\varinjlim E_\alpha, \varinjlim \mathfrak{M}_\alpha)$ is a measurable space. If $(\lambda_\alpha)_{\alpha \in I}$ is a direct system of measures with values in a complete abelian group, if $\lambda = \varinjlim \lambda_\alpha$ is the direct limit of the family (λ_α), and if $(E_\alpha, \mathfrak{M}_\alpha, \lambda_\alpha)$ is a direct system of measure (resp. probability) spaces, then it is shown that the direct limit $(E, \mathfrak{M}, \lambda) = (\varinjlim E_\alpha, \varinjlim \mathfrak{M}_\alpha, \varinjlim \lambda_\alpha)$ is a measure (resp. probability) space. Further papers will be devoted to the applications of these direct limits in the measure (resp. probability) theory.

Introduction

The theory of projective or inverse limits of measure (resp. probability) spaces is well known. In a recent paper [1] the reader will find an ample bibliography on this subject.

Among the papers devoted to the study of direct systems of measure algebras and direct limits of measure algebras we mention particularly these of J. R. Choksi [2] and C. L. Sheffer [3].

In the present paper we consider the problem of the direct limit of a direct system of measure spaces from a point of view different from those cited above. To do this we utilise the notion of correspondence and extension of correspondences to the sets of subsets as follows.

Received May 14, 1971.
AMS 1970 subject classifications: Primary 60G05, 46G99; Secondary 60G45, 28A40.
Key words and phrases: Direct limits, systems of probability spaces, measurable spaces, measures with values in abelian groups.

Let I be a right directed preordered set, and let $(E_\alpha)_{\alpha\in I}$ be a family of sets indexed by I. For each pair (α, β) of elements of I, $\alpha \leqslant \beta$, let $\Gamma_{\beta\alpha}$ be a correspondence (cf. [4, Chap. II, Section 3, Definition 2]) between E_α and E_β.

Section 1 is devoted to the definition of the direct limit of the family $(E_\alpha)_{\alpha\in I}$, with respect to the family of correspondence. $(\Gamma_{\beta\alpha})$.

If $G = \bigcup_{\alpha\in I} E_\alpha \times \{\alpha\}$ is the sum of the family (E_α), if $\mathfrak{M}_\alpha$ is a σ-algebra in E_α for each $\alpha \in I$, if $M = \bigcup_{\alpha\in I} \mathfrak{M}_\alpha \times \{\{\alpha\}\}$ is the *sum* of the family $(\mathfrak{M}_\alpha)_{\alpha\in I}$, then (cf. Prop. 2, Section 1) M is a σ-algebra in G.

In Section 1 we also define the direct system $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$, and the direct limit $\hat{E} = \varinjlim \mathfrak{P}(E_\alpha)$, as follows.

Let $\hat{G} = \bigcup_{\alpha\in I} \mathfrak{P}(E_\alpha) \times \{\{\alpha\}\}$ be the *sum* of the family $(\mathfrak{P}(E_\alpha))$ and let $\mathfrak{P}(G) = \mathfrak{P}(\bigcup_{\alpha\in I} E_\alpha \times \{\alpha\})$ be the set of subsets of $G = \bigcup_{\alpha\in I} E_\alpha \times \{\alpha\}$. If $E = G/R = \varinjlim E_\alpha$ is the direct limit of the family (E_α) with respect to the family of correspondences $(\Gamma_{\beta\alpha})$, ϕ is the canonical mapping of G onto E, $\hat{\phi}$ is the extension [5, Section 1, No. 2] of ϕ to the sets of subsets, $\hat{R}$ is the equivalence relation on $\mathfrak{P}(G)$ associated with $\hat{\phi}$, $\hat{\psi}$ is the canonical mapping of $\mathfrak{P}(G)$ onto $\mathfrak{P}(G)/\hat{R}$, then we define the direct limit $\hat{E} = \varinjlim \mathfrak{P}(E_\alpha) = \hat{G}/\hat{R}_G$ of the direct system $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ where $\hat{R}_{\hat{G}}$ is an equivalence relation in $\hat{G}$ *induced* by $\hat{R}$ on $\hat{G}$.

We suppose the correspondences $\Gamma_{\beta\alpha}$ such that $\Gamma_{\beta\alpha}\langle X_\alpha\rangle \in \mathfrak{M}_\beta$ whenever $\alpha \leqslant \beta$, i.e., $\hat{\Gamma}_{\beta\alpha}\langle\mathfrak{M}_\alpha\rangle \subset \mathfrak{M}_\beta$. Then, in Section 2, we show that the direct limit $\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$ with respect to the family $(\hat{\Gamma}_{\beta\alpha})$ is a σ-algebra in $E = \varinjlim E_\alpha$.

More precisely, if $(E_\alpha, \mathfrak{M}_\alpha)$ is a family of measurable spaces, then $(E, \mathfrak{M}) = (\varinjlim E_\alpha, \varinjlim \mathfrak{M}_\alpha)$ is a measurable space.

It should be noted that the direct system $(\mathfrak{M}_\alpha)$ is defined by means of the extensions of the correspondences $(\Gamma_{\beta\alpha})$ to the sets of subsets, i.e. $\mathfrak{M} = \varinjlim(\mathfrak{M}_\alpha, \hat{\Gamma}_{\beta\alpha})$. Section 3 is devoted to the definition of the direct limits of measure spaces. Under the hypotheses of Section 1 and Section 2, we define a direct system of additive mappings of the direct system $(\mathfrak{M}_\alpha, \hat{\Gamma}_{\beta\alpha})$ into a direct system $(F_\alpha, f_{\beta\alpha})$ of a family of abelian groups $(F_\alpha)_{\alpha\in I}$.

Under these conditions (cf. Section 3, Theorem 1), there exists a unique mapping μ of $\mathfrak{M}$ into $\mathfrak{F} = \varinjlim(F_\alpha)$. Moreover, μ is an additive mapping.

In Section 4 we define the countable additive mapping $\lambda = \varinjlim \lambda_\alpha$, where, for each $\alpha \in I$, λ_α is a countable additive mapping of $\mathfrak{M}_\alpha$ into an abelian group F_α, such that $F_\alpha = F$, when F is a complete abelian group.

More precisely, (cf. Section 4, Theorem 2), if, for each $\alpha \in I$, λ_α is a measure on $\mathfrak{M}_\alpha$ with values in F, (λ_α) is a directed system of measures relative to I with values in the complete abelian group F, then the direct limit $\lambda = \varinjlim \lambda_\alpha$ is a measure on $E = G/R$, with values in F. Hence we have

THEOREM 3 (cf. Section 4). *If $(E_\alpha, \mathfrak{M}_\alpha, \lambda_\alpha)$ is a direct system of measure spaces, then $(E, \mathfrak{M}, \lambda) = (\varinjlim E_\alpha, \varinjlim \mathfrak{M}_\alpha, \lim \lambda_\alpha)$ is a direct measure space. In particular,* *if* $(E_\alpha, \mathfrak{M}_\alpha, \lambda_\alpha)$ *is a family of probability spaces, then* $(\varinjlim E_\alpha, \varinjlim \mathfrak{M}_\alpha, \varinjlim \lambda_\alpha)$ *is a probability space.*

1. Extension of the Direct System $(E_\alpha, \Gamma_{\alpha\beta})$

1. *The Direct System* $(E_\alpha, \Gamma_{\beta\alpha})$

Let I be a (right) *directed* preordered set, and let $(E_\alpha)_{\alpha \in I}$ be a family of sets indexed by I. For each pair (α, β) of elements of I such that $\alpha \leqslant \beta$, let $\Gamma_{\beta\alpha}$ be a correspondence [4, Chap. II, Section 3, Definition 2] *between the sets* E_α *and* E_β. We suppose that the $\Gamma_{\beta\alpha}$ satisfy the following conditions:

(LI$_{\mathrm{I}}$) $\{\alpha \leqslant \beta \leqslant \gamma\} \Rightarrow \{\Gamma_{\gamma\beta} \circ \Gamma_{\beta\alpha} = \Gamma_{\gamma\alpha}\}$,

(LI$_{\mathrm{II}}$) For each $\alpha \in I$, $\Gamma_{\alpha\alpha}$ is the identity correspondence of E_α.

On the other hand, let G be the ***sum*** [4, Chap. II, Section 4, No. 8, Definition 8] of the family of sets $(E_\alpha)_{\alpha \in I}$, i.e., $G = \bigcup_{\alpha \in I} E_\alpha \times \{\alpha\}$. For any $x \in G$, there exists a unique element $\alpha \in I$, denoted by $\lambda(x)$, such that $x \in E_\alpha$.

Let $R\{x, y\}$ be the following relation between two elements of G:

$$\langle\!\langle \exists \gamma \in I \rightsquigarrow \gamma \geqslant \alpha = \lambda(x), \gamma \geqslant \beta = \lambda(y) \text{ and } \Gamma_{\gamma\alpha}\langle\{x\}\rangle = \Gamma_{\gamma\beta}\langle\{y\}\rangle\rangle\!\rangle.$$

It is easy to prove that R is an equivalence relation on G.[1]

Indeed R is reflexive and symmetric on G. On the other hand, let $x \in E_\alpha$, $y \in E_\gamma$, $z \in E_\gamma$ and suppose that there exist $\lambda \in I$ such that $\lambda \geqslant \alpha$, $\lambda \geqslant \beta$, and $\Gamma_{\lambda\alpha}\langle\{x\}\rangle = \Gamma_{\lambda\beta}\langle\{y\}\rangle$, and $\mu \in I$ such that $\mu \geqslant \beta$, $\mu \geqslant \gamma$ and $\Gamma_{\mu\beta}\langle\{y\}\rangle = \Gamma_{\mu\gamma}\langle\{z\}\rangle$. Since I is a directed set, $\exists \nu \in I \rightsquigarrow \nu \geqslant \lambda$ and $\nu \geqslant \mu$; hence, by (LI$_{\mathrm{I}}$), we have:

$$\begin{aligned} \Gamma_{\nu\alpha}\langle\{x\}\rangle &= \Gamma_{\nu\lambda}\langle\Gamma_{\lambda\alpha}\langle\{x\}\rangle\rangle = \Gamma_{\nu\lambda}\langle\Gamma_{\lambda\beta}\langle\{y\}\rangle\rangle = \Gamma_{\nu\beta}\langle\{y\}\rangle \\ &= \Gamma_{\nu\mu}\langle\Gamma_{\mu\beta}\langle\{y\}\rangle = \Gamma_{\nu\mu}\langle\Gamma_{\mu\gamma}\langle\{z\}\rangle\rangle = \Gamma_{\nu\gamma}\langle\{z\}\rangle. \end{aligned}$$

Therefore R is transitive.

Let $E = G/R$ be the quotient set of G by R. E is called the *extension direct limit*[2] of the family $(E_\alpha)_{\alpha \in I}$, with respect to the family of correspondences $(\Gamma_{\beta\alpha})$. Thus $E = \varinjlim(E_\alpha, \Gamma_{\beta\alpha}) = \varinjlim(E_\alpha) = G/R$. If ϕ is the canonical mapping of G onto G/R, we denote by ϕ_α the restriction of ϕ to E_α.

[1] Let us recall [4, Chap. II, Section 4, No. 8] that we identify the E_α with their canonical images in G and, for each $x \in G$, we denote by $\lambda(x)$ the unique index $\alpha \in I$ such that $x \in E_\alpha$.

[2] On the direct systems and direct limits with respect to families of mappings cf. [6, Section 7]. Notation and terminology of this paper follows [6].

2. *The Measurable Space* (G, M)

PROPOSITION 1. *If $\mathfrak{M}_\alpha$ is a σ-algebra in E_α, then $\mathfrak{M}_\alpha \times \{\{\alpha\}\}$ is a σ-algebra in $E_\alpha \times \{\alpha\}$.*

Proof. We have $(E_\alpha \in \mathfrak{M}_\alpha) \Rightarrow (\{E_\alpha \times \{\alpha\} \in \mathfrak{M}_\alpha \times \{\{\alpha\}\}\}$, since $E_\alpha \in \mathfrak{M}_\alpha) \Rightarrow (\{E_\alpha\} \subset \mathfrak{M}_\alpha) \Rightarrow (\{E_\alpha\} \times \{\{\alpha\}\} \subset \mathfrak{M}_\alpha \times \{\{\alpha\}\})$, and $\{E_\alpha\} \times \{\{\alpha\}\} = \{E_\alpha \times \{\alpha\}\}$. Whence $(\{E_\alpha \times \{\alpha\}\} \subset \mathfrak{M}_\alpha \times \{\{\alpha\}\}) \Rightarrow (E_\alpha \times \{\alpha\} \in \mathfrak{M}_\alpha \times \{\{\alpha\}\}) \Rightarrow (\mathfrak{M}_\alpha \times \{\{\alpha\}\}$ satisfies $(\mathrm{M_I})$). [The conditions $\mathrm{M_I}$, $\mathrm{M_{II}}$, $\mathrm{M_{III}}$ are recalled in the next footnote below.]

On the other hand, $(X_\alpha \in \mathfrak{M}_\alpha) \Rightarrow (\{X_\alpha\} \subset \mathfrak{M}_\alpha) \Rightarrow \{X_\alpha\} \times \{\{\alpha\}\} \subset \mathfrak{M}_\alpha \times \{\{\alpha\}\} \Rightarrow \{X_\alpha\} \times \{\{\alpha\}\} = \{X_\alpha \times \{\alpha\}\} \subset \mathfrak{M}_\alpha \times \{\{\alpha\}\} \Rightarrow X_\alpha \times \{\alpha\} \in \mathfrak{M}_\alpha \times \{\{\alpha\}\}$, and

$$\complement_{E_\alpha \times \{\alpha\}} (X_\alpha \times \{\alpha\}) = \complement_{E_\alpha} X_\alpha \times \{\alpha\};$$

indeed,

$$\begin{aligned} x_\alpha' \in \complement_{E_\alpha \times \{\alpha\}} & (X_\alpha \times \{\alpha\}) \\ & \Rightarrow x_\alpha' \in E_\alpha \times \{\alpha\} - X_\alpha \times \{\alpha\} \Rightarrow x_\alpha' \in E_\alpha \times \{\alpha\} \wedge x_\alpha' \notin X_\alpha \times \{\alpha\} \\ & \Rightarrow x_\alpha' = (x_\alpha, \alpha),\ x_\alpha \in E_\alpha,\ x_\alpha \notin X_\alpha;\ \alpha \in \{\alpha\} \Rightarrow x_\alpha \in \complement_{E_\alpha} X_\alpha,\ \alpha \in \{\alpha\} \\ & \Rightarrow x_\alpha' \in \Big(\complement_{E_\alpha} X_\alpha\Big) \times \{\alpha\} \Rightarrow \complement_{E_\alpha \times \{\alpha\}} (X_\alpha \times \{\alpha\}) \subset \Big(\complement_{E_\alpha} X_\alpha\Big) \times \{\alpha\}. \end{aligned}$$

Conversely,

$$\begin{aligned} x_\alpha' \in \Big(\complement_{E_\alpha} X_\alpha\Big) \times \{\alpha\} \Rightarrow x_\alpha' = (x_\alpha, \alpha) \in \Big(\complement_{E_\alpha} X_\alpha\Big) \times \{\alpha\} \Rightarrow x_\alpha \in \complement_{E_\alpha} X_\alpha; \\ \alpha \in \{\alpha\} \Rightarrow x_\alpha \in E_\alpha - X_\alpha,\ \alpha \in \{\alpha\} \Rightarrow x_\alpha \in E_\alpha \wedge x_\alpha \notin X_\alpha, \\ \alpha \in \{\alpha\} \Rightarrow (x_\alpha, \alpha) \in E_\alpha \times \{\alpha\} \wedge (x_\alpha, \alpha) \notin X_\alpha \times \{\alpha\} \\ \Rightarrow x_\alpha' \in \complement_{E_\alpha \times \{\alpha\}} (X_\alpha \times \{\alpha\}). \\ \Rightarrow \complement_{E_\alpha} X_\alpha \times \{\alpha\} \subset \complement_{E_\alpha \times \{\alpha\}} (X_\alpha \times \{\alpha\}). \end{aligned}$$

Therefore,

$$\complement_{E_\alpha \times \{\alpha\}} (X_\alpha \times \{\alpha\}) = \Big(\complement_{E_\alpha} X_\alpha\Big) \times \{\alpha\}$$

But

$$X_\alpha \in \mathfrak{M}_\alpha \Rightarrow \complement_{E_\alpha} X_\alpha \in \mathfrak{M}_\alpha \Rightarrow \left(\complement_{E_\alpha} X_\alpha\right) \times \{\alpha\} = \complement_{E_\alpha \times \{\alpha\}} (X_\alpha \times \{\alpha\}) \in \mathfrak{M}_\alpha \times \{\{\alpha\}\}.$$

$$\Rightarrow \mathfrak{M}_\alpha \times \{\{\alpha\}\} \text{ satisfies } (\mathrm{M_{II}}).$$

Suppose $(X_\alpha{}^\nu)_{\nu\in\mathbf{N}} \subset \mathfrak{M}_\alpha$, then

$$\bigcup_{\nu\in N} \{X_\alpha{}^\nu \times \{\alpha\}\} = \left\{\bigcup_{\nu\in\mathbf{N}} (X_\alpha{}^\nu \times \{\alpha\})\right\} \subset \mathfrak{M}_\alpha \times \{\{\alpha\}\} \Rightarrow \bigcup_{\nu\in\mathbf{N}} (E_\alpha \times \{\alpha\}) \in \mathfrak{M}_\alpha \times \{\{\alpha\}\}.$$

$$\Rightarrow \mathfrak{M}_\alpha \times \{\{\alpha\}\} \text{ satisfies } (\mathrm{M_{III}}).$$

Thus $\mathfrak{M}_\alpha \times \{\{\alpha\}\}$ is a σ-algebra in $E_\alpha \times \{\alpha\}$.

PROPOSITION 2. *Let $(E_\alpha, \Gamma_{\beta\alpha})$ be a direct system of sets as defined in No. 1, above.*

Let $G = \bigcup_{\alpha\in I} E_\alpha \times \{\alpha\}$ be the sum of the family $(E_\alpha)_{\alpha\in I}$; let $\mathfrak{M}_\alpha$ be a σ-algebra in E_α for each $\alpha \in I$ and let $M = \bigcup_{\alpha\in I} \mathfrak{M}_\alpha \times \{\{\alpha\}\}$ be the sum of the family $(\mathfrak{M}_\alpha)_{\alpha\in I}$.

Then, M is a σ-algebra in G.

Proof.[3] We have

$$(E_\alpha \in \mathfrak{M}_\alpha) \Rightarrow (E_\alpha \times \{\alpha\} \in \mathfrak{M}_\alpha \times \{\{\alpha\}\} \text{ by virtue of the Proposition 1})$$

$$\Rightarrow \{E_\alpha \times \{\alpha\}\} \subset \mathfrak{M}_\alpha \times \{\{\alpha\}\}$$

$$\Rightarrow \bigcup_{\alpha\in I} \{E_\alpha \times \{\alpha\}\} \subset \bigcup_{\alpha\in I} (\mathfrak{M}_\alpha \times \{\{\alpha\}\}) = M.$$

But

$$\bigcup_{\alpha\in I} \{E_\alpha \times \{\alpha\}\} = \left\{\bigcup_{\alpha\in I} E_\alpha \times \{\alpha\}\right\} \Rightarrow G = \bigcup_{\alpha\in I} E_\alpha \times \{\alpha\} \in \bigcup_{\alpha\in I} (\mathfrak{M}_\alpha \times \{\{\alpha\}\}) = M$$

$$\Rightarrow (M \text{ satisfies } (M_{\mathrm{I}})).$$

[3] Let us recall that a collection $\mathfrak{M}$ of subsets of a set E is said to be a σ-algebra in E if $\mathfrak{M}$ satisfies the following axioms:

(M_{I}) $E \in \mathfrak{M}$;

(M_{II}) $A \in \mathfrak{M} \Rightarrow \complement A \in \mathfrak{M}$, where $\complement A$ is the complement of A relative to E;

(M_{III}) $A_\nu \in \mathfrak{M}$, $\forall \nu \in \mathbf{N} = \{0, 1, 2,\ldots\} \Rightarrow \bigcup_{\nu\in N} A_\nu \in \mathfrak{M}$.

On the other hand,

$$(X \in M) \Rightarrow (\exists \alpha(X) \in I \rightsquigarrow X \in \mathfrak{M}_{\alpha(X)} \Rightarrow X \times \{_{\alpha(X)}\} \in \mathfrak{M}_{\alpha(X)} \times \{\{_{\alpha(X)}\}\}$$

$$\Rightarrow X \subset E_{\alpha(X)} \times \{\alpha(X)\} \Rightarrow X \subset \bigcup_{\alpha \in I} E_\alpha \times \{\alpha\} = G$$

$$\Rightarrow \complement_G X = (E_{\alpha(X)} \times \{\alpha(X)\} - X) \cup \Big(\bigcup_{\beta \neq \alpha} E_\beta \times \{\beta\}\Big)$$

$$= \Big(\complement_{E_{\alpha(X)} \times \{\alpha(X)\}} X\Big) \cup \Big(\bigcup_{\beta \neq \alpha} E_\beta \times \{\beta\}\Big).$$

But

$$\complement_{E_{\alpha(X)} \times \{\alpha(X)\}} (X) \in \mathfrak{M}_{\alpha(X)} \times \{\{\alpha(X)\}\}$$

$$\Rightarrow \complement_{E_{\alpha(X)} \times \{\alpha(X)\}} (X) \subset E_{\alpha\{X\}} \times \{\alpha(X)\}$$

$$\Rightarrow \Big(\complement_{E_{\alpha(X)} \times \{\alpha(X)\}} X\Big) \cup \Big(\bigcup_{\beta \neq \alpha} (E_\beta \times \{\beta\})\Big) \subset \bigcup_{\alpha \in I} (E_\alpha \times \{\alpha\})$$

and

$$\Big\{\complement_{E_{\alpha(X)} \times \{\alpha(X)\}} X\Big\} \subset \mathfrak{M}_{\alpha(X)} \times \{\{\alpha(X)\}\} \Rightarrow \Big\{\complement_{E_{\alpha\{X\}} \times \{\alpha(X)\}} X\Big\} \cup \Big(\Big\{\bigcup_{\beta \neq \alpha} E_\beta \times \{\beta\}\Big\}\Big) \subset M$$

$$\Rightarrow \complement_G X \in M \Rightarrow M \text{ satisfies } (\mathrm{M_{II}}).$$

Finally,

$$(X_\nu \in M, \forall \nu \in \mathbf{N}) \Rightarrow (\exists \alpha_\nu \in I \rightsquigarrow X_\nu \in \mathfrak{M}_{\alpha_\nu} \times \{\{\alpha_\nu\}\}$$

$$\Rightarrow \bigcup_{\nu \in \mathbf{N}} \{X_\nu\} = \Big\{\bigcup_{\nu \in \mathbf{N}} X_\nu\Big\} \subset \bigcup_{\nu \in \mathbf{N}} (\mathfrak{M}_{\alpha_\nu} \times \{\{\alpha_\nu\}\})$$

$$\Rightarrow \Big(\bigcup_{\nu \in \mathbf{N}} X_\nu \in \bigcup_{\nu \in \mathbf{N}} (\mathfrak{M}_{\alpha_\nu} \times \{\{\alpha_\nu\}\}) \subset M\Big)$$

$$\Rightarrow \Big(\bigcup_{\nu \in \mathbf{N}} X_\nu \in M\Big) \Rightarrow M \text{ satisfies } (\mathrm{M_{III}}).$$

3. *The Direct System* $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$

Let $(E_\alpha, \Gamma_{\beta\alpha})$ be a direct system of sets relative to the directed set I, as defined in Section 1, No. 1, above and let $\hat{\Gamma}_{\beta\alpha}$ be the extension of $\Gamma_{\beta\alpha}$ to the set of subsets, defined by

$$\hat{\Gamma}_{\beta\alpha}(X_\alpha) = \Gamma_{\beta\alpha}\langle X_\alpha\rangle \in \mathfrak{P}(E_\beta), \qquad \forall X_\alpha \in \mathfrak{P}(E_\alpha) \qquad \text{for} \quad \alpha \leqslant \beta. \tag{1}$$

We have

$$\hat{\Gamma}_{\beta\alpha} \in \hom(\mathfrak{P}(E_\alpha), \mathfrak{P}(E_\beta)),$$

where $\hom(\mathfrak{P}(E_\alpha), \mathfrak{P}(E_\beta)$ is the set of all mappings of $\mathfrak{P}(E_\alpha)$ into $\mathfrak{P}(E_\beta)$. For $\alpha \leqslant \beta \leqslant \gamma$, we have

$$E_\alpha \xmapsto{\Gamma_{\beta\alpha}} E_\beta \xmapsto{\Gamma_{\gamma\beta}} E_\gamma \Rightarrow \mathfrak{P}(E_\alpha) \xmapsto{\hat{\Gamma}_{\beta\alpha}} \mathfrak{P}(E_\beta) \xmapsto{\hat{\Gamma}_{\gamma\beta}} \mathfrak{P}(E_\gamma);$$
$$\Gamma_{\gamma\beta} \circ \Gamma_{\beta\alpha} = \Gamma\gamma\alpha \Rightarrow \hat{\Gamma}_{\gamma\beta} \circ \hat{\Gamma}_{\beta\alpha} = \hat{\Gamma}_{\gamma\alpha}$$

and

$$\{\Gamma_{\alpha\alpha} = I_{E\alpha} = (\Delta_\alpha, E_\alpha, E_\alpha)\} \Rightarrow \{\hat{\Gamma}_{\alpha\alpha} = I_{\mathfrak{P}(E_\alpha)} = (\hat{\Delta}_\alpha, \mathfrak{P}(E_\alpha), \mathfrak{P}(E_\alpha)),$$

where Δ_α (resp. $\hat{\Delta}_\alpha$) is the diagonal of $E_\alpha \times E_\alpha(\mathfrak{P}(E_\alpha) \times \mathfrak{P}(E_\alpha))$. Let

$$\hat{G} = \bigcup_{\alpha \in I} \mathfrak{P}(E_\alpha) \times \{\{\alpha\}\},$$

then $\forall X \in \hat{G}$, there exists a unique $\alpha = \alpha(X)$ such that $X \in \mathfrak{P}(E_\alpha) \times \{\{\alpha\}\}$.

On the other hand, let $\hat{R}_G\{X, Y\}$ be a relation between two elements X, Y of $\hat{G}$ defined as follows:

$$\langle\!\langle \exists\gamma \in I \rightsquigarrow \gamma \geqslant \alpha = \lambda(X), \gamma \geqslant \beta = \lambda(Y) \text{ and } (\hat{\Gamma}_{\gamma\alpha}(X) = \hat{\Gamma}_{\gamma\beta}(Y)\rangle\!\rangle.$$

It is easy to see that $\hat{R}_G$ is an equivalence relation on $\hat{G}$.

Let $\hat{\psi}_G$ be the canonical mapping of $\hat{G}$ onto $\hat{G}/\hat{R}_G$.

We have

$$\hat{R}_G\{X, Y\} \Leftrightarrow \{\hat{\psi}_G(X) = \hat{\psi}_G(Y)\} \Leftrightarrow \{\hat{\Gamma}_{\gamma\alpha}(X) = \hat{\Gamma}_{\gamma\beta}(Y)\},$$
$$\text{for} \quad \gamma \geqslant \alpha = \lambda(X), \quad \gamma \geqslant \beta = \lambda(Y);$$

$$\begin{aligned} X \in \mathfrak{P}(E_\alpha) &\times \{\{\alpha\}\},\ Y \in \mathfrak{P}(E_\beta) \times \{\{\beta\}\} \\ &\Leftrightarrow \{\exists\gamma \in I \rightsquigarrow \gamma \geqslant \alpha = \lambda(X), \gamma \geqslant \beta = \lambda(Y), \Gamma_{\gamma\alpha}\langle X\rangle = \Gamma_{\gamma\beta}\langle Y\rangle\} \\ &\Leftrightarrow \{\forall x \in X, \exists y \in Y \text{ and } \forall y \in Y, \exists x \in X \rightsquigarrow \phi\langle\{x\}\rangle = \phi\langle\{y\}\rangle\} \\ &\Leftrightarrow \{\phi\langle X\rangle = \phi\langle Y\rangle\}, \qquad \text{for} \quad X \in \mathfrak{P}(E_\alpha), \quad Y \in \mathfrak{P}(E_\beta), \end{aligned}$$

where ϕ is the canonical mapping of G onto $G/R = E = \varinjlim E_\alpha$, and where by identification we set $X \in \mathfrak{P}(E_\alpha)$, $Y \in \mathfrak{P}(E_\beta)$. Let $\hat{\phi}$ be the (canonical) extension of ϕ to the sets of subsets; $\hat{\phi}$ is a surjective mapping of $\mathfrak{P}(G) = \mathfrak{P}(\bigcup_{\alpha \in I} E_\alpha \times \{\alpha\})$ onto $\mathfrak{P}(G/R) = \mathfrak{P}(\bigcup_{\alpha \in I} E_\alpha \times \{\alpha\}/R)$.

Suppose that $\hat{R}$ is the equivalence relation on $\mathfrak{P}(G)$ associated with $\hat{\phi}$, i.e., (cf. [5, Section 1, No. 2]):

$$\hat{R}\{X, Y\} \Leftrightarrow \hat{\psi}(X) = \hat{\psi}(Y) \Leftrightarrow \hat{\phi}(X) = \hat{\phi}(Y),$$

where X and Y are elements of $\mathfrak{P}(G)$, and $\hat{\psi}$ is the canonical mapping of $\mathfrak{P}(G)$ onto $\mathfrak{P}(G)/\hat{R}$. Under these conditions, all theorems and propositions established in [5] are true if we replace the set E everywhere by the set G.

On the other hand, we have $\hat{G} \subset \mathfrak{P}(G)$; therefore, $\hat{R}_G$ is identical to the equivalence relation induced[4] by $\hat{R}$ on $\hat{G}$. The equivalence classes with respect to $\hat{R}_G$ are the *traces* on $\hat{G}$ of the equivalence classes with respect to $\hat{R}$ which *meet* $\hat{G}$. Hence, it follows from Fig. 1 that the injection j_G is *compatible* with the equivalence relations $\hat{R}_G$ and $\hat{R}$, and the mapping h of $\hat{G}/\hat{R}_G$ into $\mathfrak{P}(G)/\hat{R}$ induced by j_G on passing to the quotients with respect to $\hat{R}_G$ and $\hat{R}$, is an *injective mapping* of $\hat{G}/\hat{R}_G$ into $\mathfrak{P}(G)/\hat{R}$ (cf. [4, Chap. II, Section 6, No. 6]).

$$\begin{array}{ccc} \hat{G} & \xrightarrow{j_G} & \mathfrak{P}(G) \\ {\scriptstyle \hat{\psi}_G}\downarrow & & \downarrow{\scriptstyle \hat{\psi}} \\ \dfrac{\hat{G}}{\hat{R}_G} & \xrightarrow[h]{} & \dfrac{\mathfrak{P}(G)}{\hat{R}} \end{array}$$

FIGURE 1

Remark 1. If $\hat{k}$ is the canonical bijection of $\hat{G}/\hat{R}_G$ onto $h\langle\hat{G}/\hat{R}_G\rangle$, we have

$$\frac{\hat{G}}{\hat{R}_G} \xrightarrow{\hat{k}} h\left\langle\frac{\hat{G}}{\hat{R}_G}\right\rangle = \hat{\psi}\langle\hat{G}\rangle.$$

Therefore one can *identify* $\hat{G}/\hat{R}_G$ with $\hat{\psi}\langle\hat{G}\rangle$.

Let $(\mathfrak{P}(E_\alpha), \hat{f}_{\beta\alpha})$ be the direct system of sets relative to the directed set I and let $\hat{E} = \varinjlim(\mathfrak{P}(E_\alpha), \hat{f}_{\beta\alpha}) = \hat{G}/\hat{R}_G$ be the direct limit of the family $(\mathfrak{P}(E_\alpha))_{\alpha \in I}$ with respect to the family of mappings $(\hat{f}_{\beta\alpha})$.

We have (cf. [4, Section 7, No. 6, Prop. 6]):

[4] Compare [4, Chap. II, No. 6].

PROPOSITION 3. *For each $\alpha \in I$, let $\hat{u}_\alpha$ be a mapping of $\mathfrak{P}(E_\alpha)$ into a set F, such that $\hat{h}_\beta \circ \hat{\Gamma}_{\beta\alpha} = \hat{u}_\alpha$ for $\alpha \leqslant \beta$, and let $\hat{\psi}_{\mathfrak{P}(E_\alpha)}$ be the restriction of the canonical mapping $\hat{\psi}_G$ to the set $\mathfrak{P}(E_\alpha)$. Then:*

(a) *There exists a unique mapping $\hat{u}$ of $\hat{E}$ into F such that $\hat{u}_\alpha = \hat{u} \circ \hat{\psi}_{\mathfrak{P}(E_\alpha)}$ for all $\alpha \in I$.*

(b) *$\hat{u}$ is a surjection if and only if*

$$F = \bigcup_{\alpha \in I} \hat{u}_\alpha(\mathfrak{P}(E_\alpha)).$$

(c) *$\hat{u}$ is an injection if and only if, for each $\alpha \in I$, the relations $X \in \mathfrak{P}(E_\alpha)$, $Y \in \mathfrak{P}(E_\alpha)$, $\hat{u}_\alpha(X) = \hat{u}_\alpha(Y)$ imply that there exists $\beta \geqslant \alpha$ such that $\hat{\Gamma}_{\beta\alpha}(X) = \hat{\Gamma}_{\beta\alpha}(Y)$.*

The direct system $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{(\beta\alpha)})$ is called the extension of the direct system $(E_\alpha, \Gamma_{\beta\alpha})$ to the sets of subsets.

Remark 2. For $\alpha \leqslant \beta$, we have:

$$\left\{\mathfrak{P}(E_\alpha) \xmapsto{\hat{\Gamma}_{\beta\alpha}} \mathfrak{P}(E_\beta) \xmapsto{\hat{\psi}_{\mathfrak{P}(E_\beta)}} \frac{\mathfrak{P}(G)}{\hat{R}}\right\} \Rightarrow \{\hat{\psi}_{\mathfrak{P}(E_\alpha)} = \hat{\psi}_{\mathfrak{P}(E_\beta)} \circ \hat{\Gamma}_{\beta\alpha}\},$$

where $\hat{\psi}_{\mathfrak{P}(E_\alpha)}$ (resp. $\hat{\psi}_{\mathfrak{P}(E_\beta)}$)

is the restriction to $\mathfrak{P}(E_\alpha)(\mathfrak{P}(E_\beta))$ of the canonical mapping $\hat{\psi}$. As $\hat{\psi}$ is a surjection, then by (b), where we set $\hat{u}_\alpha = \hat{\psi}_\alpha$, we have

$$\frac{\mathfrak{P}(G)}{\hat{R}} = \bigcup_{\alpha \in I} \hat{\psi}_{\mathfrak{P}(E_\alpha)}(\mathfrak{P}(E_\alpha)).$$

PROPOSITION 4. *Let $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ be the direct system which is the extension of the direct system $(E_\alpha, \Gamma_{\beta\alpha})$ to the sets of subsets; let $(F_\alpha, f_{\beta\alpha})$ be a direct system relative to the same index set I, let $\hat{E} = \varinjlim(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$, let $F = \varinjlim(F_\alpha, f_{\beta\alpha})$, and for each $\alpha \in I$ let f_α be the canonical mapping of F_α into F such that whenever $\alpha \leqslant \beta$ the diagram*

$$\begin{tikzcd}
(E_\alpha) \arrow[r, mapsto, "\hat{u}_\alpha"] \arrow[d, mapsto, "\hat{\Gamma}_{\beta\alpha}"'] & F_\alpha \arrow[d, mapsto, "f_{\beta}"] \\
(E_\beta) \arrow[r, mapsto, "\hat{u}_\beta"'] & F_\beta
\end{tikzcd}$$

FIGURE 2

is commutative.

Then there exists a unique mapping $\hat{u}$ of $\hat{E}$ into F such that, for each $\alpha \in I$, the diagram

$$\begin{tikzcd}
\mathfrak{P}(E_\alpha) \arrow[r, mapsto, "\hat{u}_\alpha"] \arrow[d, mapsto, "\hat{\varphi}_{\mathfrak{P}(E_\alpha)}"'] & F_\alpha \arrow[d, mapsto, "f_\alpha"] \\
\hat{E} \arrow[r, mapsto, "\hat{u}"'] & F
\end{tikzcd}$$

FIGURE 3

is commutative.

For proof, cf. [6, Section 7, No. 6, Corollary 1 of Proposition 6]. The family $(\hat{u}_\alpha)_{\alpha \in I}$ is a *direct system of mappings of* $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ into $(F_\alpha, f_{\beta\alpha})$, and $\hat{u}$ is the *direct limit* of the family $(\hat{u}_\alpha)$; we set $\hat{u} = \varinjlim \hat{u}_\alpha$.

PROPOSITION 5. *Let $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ be the directed system which is the extension of the direct system $(E_\alpha, \Gamma_{\beta\alpha})$; let $(F_\alpha, f_{\alpha\beta})$ be a direct system of sets relative to the same index set I, and for each $\alpha \in I$, let $\hat{u}_\alpha$ be a mapping of $\mathfrak{P}(E_\alpha)$ into F_α such that $(\hat{u}_\alpha)_{\alpha \in I}$ is a direct system of mappings of $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ into $(F_\alpha, f_{\beta\alpha})$.*

Let $\hat{u} = \varinjlim \hat{u}_\alpha$ be the direct limit of the family $(\hat{u}_\alpha)$. Then, if each $\hat{u}_\alpha$ is injective (surjective) *$\hat{u}$ is injective* (surjective).

For the proof, cf. [6, Section 7, No. 6, Proposition 7].

2. DIRECT LIMITS OF MEASURABLE SPACES

1. *The σ-Algebra $\mathfrak{M}$*

THEOREM 1. *Suppose that I is a* (right) *directed preordered set, $(E_\alpha, \Gamma_{\beta\alpha})$ is a direct system of sets relative to the directed set I, with respect to the family of correspondences $(\Gamma_{\beta\alpha})$* (cf. Section 1, No. 2), *$G = \bigcup_{\alpha \in I} E_\alpha \times \{\alpha\}$ is the sum and $E = \varinjlim (E_\alpha, \Gamma_{\beta\alpha}) = G/R$ is the direct limit of the family (E_α). Let, $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ be the direct system, extension of the direct system $(E_\alpha, \Gamma_{\beta\alpha})$; let*

$$\hat{G} = \bigcup_{\alpha \in I} \mathfrak{P}(E_\alpha) \times \{\{\alpha\}\}$$

be the sum of the family $(\mathfrak{P}(E_\alpha))$, let $\hat{E} = \hat{G}/\hat{R}_G = \varinjlim (\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ be the direct limit of the family $(\mathfrak{P}(E_\alpha))$.

For each $\alpha \in I$, let $\mathfrak{M}_\alpha$ be a σ-algebra in E_α, and suppose that the correspondences $\Gamma_{\beta\alpha}$ are such that, for each $X \in \mathfrak{M}_\alpha$, we have $\hat{\Gamma}_{\beta\alpha}(X) = \Gamma_{\beta\alpha}\langle X \rangle \in \mathfrak{M}_\beta$ whenever $\alpha \leqslant \beta$, i.e., *$\hat{\Gamma}_{\beta\alpha}\langle \mathfrak{M}_\alpha \rangle \subset \mathfrak{M}_\beta$. Let $M = \bigcup_{\alpha \in I} \mathfrak{M}_\alpha \times \{\{\alpha\}\}$ be the sum and $\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$*

the direct limit of the family $(\mathfrak{M}_\alpha)_{\alpha \in I}$. *Suppose that* ϕ *is the canonical mapping of* G *onto* E; $\hat{\phi}$ *is the extension of* ϕ *to the sets of subsets*; $\hat{R}$ *is the equivalence relation on* $\mathfrak{P}(G)$ *associated with* $\hat{\phi}$; $\hat{\psi}$ *is the canonical mapping of* $\mathfrak{P}(G)$ *onto* $\mathfrak{P}(G)/\hat{R}$ *and* $\hat{g}$ *is the bijection* (*cf.* [5, No. 2, Theorem 1) *of* $\mathfrak{P}(G/R)$ *onto* $\mathfrak{P}(G)/\hat{R}$. *Under these conditions,* $\mathfrak{M}$ *is a* σ*-algebra in* $\hat{g}(E)$, *and by identification of* $\mathfrak{P}(G/R)$ *with* $\mathfrak{P}(G)/\hat{R}$, $\mathfrak{M}$ *is a* σ*-algebra in* E.

Proof. Since $\mathfrak{M}_\alpha \subset \mathfrak{P}(E_\alpha)$ for each $\alpha \in I$, then the family $(\mathfrak{M}_\alpha)_{\alpha \in I}$ is a direct system of subsets of the $\mathfrak{P}(E_\alpha)$ (cf. [6, Section 7, No. 6, p. 94]). Let $\hat{g}_{\beta\alpha}$ (for $\alpha \leqslant \beta$) be the mapping of $\mathfrak{M}_\alpha$ into $\mathfrak{M}_\beta$, the graph of which is the same as that of the restriction of $\hat{\Gamma}_{\beta\alpha}$ to $\mathfrak{M}_\alpha$. Then $(\mathfrak{M}_\alpha, \hat{g}_{\beta\alpha})$ is a direct system of sets, and Proposition 5 of Section 1, No. 3, applied to the canonical injections $\hat{j}_\alpha$ of $\mathfrak{M}_\alpha$ into $\mathfrak{P}(E_\alpha)$, allows us to identify the direct limit $\mathfrak{M} = \varinjlim(\mathfrak{M}_\alpha, \hat{g}_{\beta\alpha})$ with a subset of $\hat{E}$ by means of the injection $\hat{j} = \varinjlim \hat{j}_\alpha$; i.e. (cf. Section 1, No. 3, Remark 1):

$$\mathfrak{M} \subset \frac{\hat{G}}{\hat{R}_G} = \hat{\psi}\langle \hat{G} \rangle.$$

On the other hand, $\{M \subset \hat{G}\} \Rightarrow \{$one can identify $\mathfrak{M}$ with $\hat{\psi}\langle M \rangle\}$, i.e., if $\hat{\psi}_M$ is the canonical mapping of M onto $\mathfrak{M}$, we have $\mathfrak{M} = \hat{\psi}_M\langle M \rangle = \hat{\psi}\langle M \rangle$. But (cf. [5, Theorem 1, No. 2]) $\{\hat{\psi} = \hat{g} \circ \hat{\phi}\} \Rightarrow$

$$\mathfrak{M} = \hat{\psi}\langle M \rangle = \hat{g}\langle \hat{\phi}\langle M \rangle \rangle. \tag{1}$$

Let us now prove that $\mathfrak{M}$ is a σ-algebra in E.

We have $E = G/R = \phi\langle G \rangle$; but $G \in M$, since M is a σ-algebra in G (cf. Section 1, No. 2, Proposition 1).

Therefore, $\hat{\psi}(G) = \hat{g}(\hat{\phi}(G)) = \hat{g}(\phi\langle G \rangle) = \hat{g}(E) \in \mathfrak{M}$.

$$\Rightarrow \left\{ E \in \mathfrak{M} \text{ by identification of } \mathfrak{P}\left(\frac{G}{R}\right) \text{ with } \frac{\mathfrak{P}(G)}{\hat{R}} \right.$$

$$\left. \text{by Theorem 1, of Section 1, No. 2} \right\}$$

$$\Rightarrow \{\mathfrak{M} \text{ satisfies } (\mathrm{M_I})\}.$$

On the other hand,

$$\{X \in \mathfrak{M}\} \Rightarrow \{\exists \alpha \in I \text{ (cf. [6, Section 7, No. 5, Lemma 1])} \rightsquigarrow X = \hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha) \text{ and } X_\alpha \in \mathfrak{M}_\alpha\}$$

$$\Rightarrow \{X = \hat{\psi}(X_\alpha) = \hat{g}(\phi_\alpha\langle X_\alpha \rangle) \text{ where } \phi_\alpha \text{ is the restriction of } \phi \text{ to } E_\alpha\}$$

$$\Rightarrow \{X = g\langle\langle \phi\langle X_\alpha \rangle\rangle \text{ where } g = \hat{g}\,|_E = \text{restriction of } \hat{g} \text{ to } E\}$$

$$\Rightarrow \left\{ \complement_M X = \complement_M g\langle \phi\langle X_\alpha \rangle\rangle = g\left\langle \complement_E \left\langle \phi\langle X_\alpha \rangle \right\rangle \subset \hat{g}(E) = \hat{g}(\phi\langle G \rangle) \right\}.$$

But

$$\{G \in M\} \Rightarrow \{\hat{\phi}(G) = \phi\langle G\rangle \in \hat{\phi}\langle M\rangle\} \Rightarrow \Big\{g\Big\langle \complement_E \phi\langle X_\alpha\rangle\Big\rangle \subset \hat{g}(\hat{\phi}(G)) \in \hat{g}(\hat{\phi}\langle M\rangle)\Big\}$$

$$\Rightarrow \Big\{\complement_M X \in \hat{\phi}\langle M\rangle = \mathfrak{M}\Big\} \Rightarrow \{\mathfrak{M} \text{ satisfies } (\mathrm{M_{II}})\}.$$

Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of elements of $\mathfrak{M}$; then $\exists \alpha_n \in I \rightsquigarrow X_n = \hat{\phi}(X_{\alpha_n})$ for $X_{\alpha_n} \in \mathfrak{M}_{\alpha_n}$; whence

$$\bigcup_{n\in\mathbf{N}} X_n = \bigcup_{n\in\mathbf{N}} g\langle\phi\langle X_{\alpha_n}\rangle = g\left\langle\phi\left\langle\bigcup_{n\in\mathbf{N}} X_{\alpha_n}\right\rangle\right\rangle$$

$$= \hat{g}\left(\hat{\phi}\left(\bigcup_{n\in\mathbf{N}} X_{\alpha_n}\right) = \hat{\phi}\left(\bigcup_{n\in\mathbf{N}} X_{\alpha_n}\right) \in \mathfrak{M},\right.$$

since

$$\bigcup_{n\in\mathbf{N}} X_{\alpha_n} \in M.$$

Theorem 1 implies that $\{E = \varinjlim(E_\alpha, \Gamma_{\beta\alpha}), \mathfrak{M} = \varinjlim(\mathfrak{M}_\alpha, \hat{\Gamma}_{\beta\alpha})\} = \{(E, \mathfrak{M}) = (\varinjlim E_\alpha, \lim \mathfrak{M}_\alpha)\}$ is a measurable space. The pair $(E, \mathfrak{M})$ is called *the direct limit* of the family of measurable spaces $(E_\alpha, \mathfrak{M}_\alpha)_{\alpha\in I}$.

3. Direct Limits of Additive Mappings

1. *The Additive Mapping* $\mu = \varinjlim \mu_\alpha$

THEOREM 1. *Under the hypothesis of Theorem* 1, *Section* 2, *let moreover* $(F_\alpha)_{\alpha\in I}$ *be a family of abelian groups and let* $(F_\alpha, f_{\beta\alpha})$ *be the direct system of the family* (F_α), *relative to the family of homomorphisms* $(f_{\beta\alpha} : E_\alpha \to E_\beta)$, *whenever* $\alpha \leqslant \beta$. *Let* $\mathfrak{F} = \varinjlim(F_\alpha, f_{\beta\alpha})$ *be the direct limit abelian group of the direct system* $(F_\alpha, f_{\beta\alpha})$; *let* f_α *be the canonical mapping of* F_α *into* $\mathfrak{F}$; *let* (μ_α) *be a direct system of additive mappings of the direct system* $(\mathfrak{M}_\alpha, \hat{g}_{\beta\alpha})$ *into the direct system* $(F_\alpha, f_{\beta\alpha})$, *i.e., such that the diagram* (*Fig.* 4) *is commutative, whenever* $\alpha \leqslant \beta$.

$$\begin{tikzcd}
\mathfrak{M}_\alpha \arrow[r, mapsto, "\mu_\alpha"] \arrow[d, "\hat{g}_{\beta\alpha}"'] & F_\alpha \arrow[d, "f_{\beta\alpha}"] \\
\mathfrak{M}_\beta \arrow[r, mapsto, "\mu_\beta"'] & F_\beta
\end{tikzcd}$$

FIGURE 4

Under these conditions (cf., Section 1, No. 3, Proposition 3) *there exists a unique mapping μ of $\mathfrak{M}$ into $\mathfrak{F}$ such that the diagram* (*Fig.* 5) *is commutative.*

$$\begin{tikzcd}
\mathfrak{M}_\alpha \arrow[r, mapsto, "\mu_\alpha"] \arrow[d, mapsto, "\psi_{\mathfrak{M}\alpha}"'] & F_\alpha \arrow[d, mapsto, "f_\alpha"] \\
\mathfrak{M} \arrow[r, mapsto, "\mu"'] & \mathfrak{F}
\end{tikzcd}$$

FIGURE 5

Moreover, μ is an additive mapping.

Proof. For each $\alpha \in I$ the σ-algebra $\mathfrak{M}_\alpha$ may be considered as endowed with an internal law of composition, s_α defined by $s_\alpha(X_\alpha, Y_\alpha) = X_\alpha \cup Y_\alpha$ for any arbitrary pair (X_α, Y_α) of elements of $\mathfrak{M}_\alpha$.

In the same way, the σ-algebra $\mathfrak{M}$ is endowed with the internal law of composition s defined by $s(X, Y) = X \cup Y$. But for $X \in \mathfrak{M}$, $Y \in \mathfrak{M}$, there exists (cf. [6, Section 7, No. 5, Lemma 1]) a unique $\alpha \in I$ such that $X = \hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha)$ and $Y = \hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha)$, where $X_\alpha \in \mathfrak{M}_\alpha$, $Y_\alpha \in \mathfrak{M}_\alpha$. But

$$[\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha) = \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{X_\alpha\}\rangle, \hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha) = \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{Y_\alpha\}\rangle]$$
$$\Rightarrow [\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha) \cup \hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha) = \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{X_\alpha\}\rangle \cup \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{Y_\alpha\}\rangle$$
$$= \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{X_\alpha\} \cup \{Y_\alpha\}\rangle = \hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha \cup Y_\alpha).$$

Therefore

$$X \cup Y = \hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha \cup Y_\alpha) \in \mathfrak{M}.$$

Let us now consider for each $\alpha \in I$, an additive mapping μ_α of $\mathfrak{M}_\alpha$ into F_α, such that the diagram of Fig. 4 is commutative.

Then there exists (cf. Section 1, No. 3, Proposition 3) a unique mapping μ of $\mathfrak{M}$ into $\mathfrak{F}$ such that the diagram of Fig. 5 is commutative; i.e., such that

$$\mu \circ \hat{\psi}_{\mathfrak{M}_\alpha} = f_\alpha \circ \mu_\alpha. \tag{2}$$

Moreover, μ is an additive mapping. Indeed we have

$$\mu(\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha)) = f_\alpha(\mu_\alpha(X_\alpha)), \qquad \text{for} \quad X_\alpha \in \mathfrak{M}_\alpha.$$

Therefore

$$\{X = \hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha),\ Y = \hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha) \text{ are arbitrary and distinct elements of } \mathfrak{M}\}$$
$$\Rightarrow \mu(X \cup Y) = \mu(\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha) \cup \hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha)) = \mu(\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha \cup Y_\alpha)) = f_\alpha(\mu_\alpha(X_\alpha \cup Y_\alpha)).$$

But $\{X, Y\}$ are distinct elements of $\mathfrak{M}\} \Rightarrow \{X, Y$ are disjoint subsets of $M\} \Rightarrow \{X \cap Y = \varnothing$ is in $M\} \Rightarrow \{\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha) \cap \hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha) = \varnothing\}$.

On the other hand,

$$\begin{aligned}
&\{\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha \cap Y_\alpha) = \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{X_\alpha \cap Y_\alpha\}\rangle\}^5 \\
&\quad \Rightarrow \{\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha \cap Y_\alpha) \subset \hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha) \cap \hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha) = \varnothing\} \\
&\quad \Rightarrow \{\hat{\psi}_{\mathfrak{M}_\alpha}\langle\{X_\alpha \cap Y_\alpha\}\rangle = \varnothing\} \Rightarrow \{X_\alpha \cap Y_\alpha = \varnothing\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\{\mu_\alpha(X_\alpha \cup Y_\alpha) = \mu_\alpha(X_\alpha) + \mu_\alpha(Y_\alpha)\} \\
&\quad \Rightarrow \{\mu(X \cup Y) = f_\alpha(\mu_\alpha(X\alpha \cup Y\alpha)) \\
&\qquad\qquad = f_\alpha(\mu_\alpha(X_\alpha) + \mu_\alpha(Y_\alpha)) = f_\alpha(\mu_\alpha(X_\alpha)) + f_\alpha(\mu_\alpha(Y_\alpha)) \\
&\qquad\qquad = \mu(\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha)) + \mu(\hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha)) = \mu(X) + \mu(Y)\} \\
&\quad \Rightarrow \{\mu \text{ is an additive mapping}\}.
\end{aligned}$$

Let us set $\mu = \varinjlim \mu_\alpha$; then μ is the direct limit of the direct system of additive mappings $(\mu_\alpha)_{\alpha \in I}$.

4. The Countable Additive Mapping μ

1. *The Direct Limit of* $(F_\alpha)_{\alpha\in I}$, $F_\alpha = F$, $\forall \alpha \in I$.

Let I be a preordered directed set; $(F_\alpha)_{\alpha\in I}$ a family of abelian groups such that $F_\alpha = F$, $\forall \alpha \in I$, where F is a complete abelian group.

Then,

Proposition 1. *There exists a canonical isomorphism of the abelian group* $\varinjlim F_\alpha$ *onto* F.

Proof. Let $G_F = \bigcup_{\alpha\in I} G_\alpha$ be the sum of the family $(F_\alpha)_{\alpha\in I}$; i.e., for each $\alpha \in I$, there is a *canonical isomorphism* h_α of F onto G_α . For any $y \in F$, let us set

$$P_y = \bigcup_{\alpha\in I} \{h_\alpha(y)\}; \tag{1}$$

[5] One has $\{X_\alpha \cap Y_\alpha\} = \{X_\alpha\} \cap \{Y_\alpha\}$, whence

$$\begin{aligned}
&\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha \cap Y_\alpha) = \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{X_\alpha \cap Y_\alpha\}\rangle \\
&\quad = \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{X_\alpha\} \cap \{Y_\alpha\}\rangle \subset \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{X_\alpha\}\rangle \cap \hat{\psi}_{\mathfrak{M}_\alpha}\langle\{Y_\alpha\}\rangle = \hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha) \cap \hat{\psi}_{\mathfrak{M}_\alpha}(Y_\alpha).
\end{aligned}$$

we have

$$P_y \cap P_{y'} = \varnothing .$$
$$y \neq y'$$

Indeed,

$$\{u \in P_y \text{ and } u' \in P_{y'}\}$$
$$\Rightarrow \{(\exists\alpha)(\alpha \in I) \rightsquigarrow u = h_\alpha(y) \in G_\alpha \text{ and } (\exists\alpha')(\alpha' \in I) \rightsquigarrow u' = h'_{\alpha'}(y') \in G_{\alpha'}\}.$$

On the other hand, if $P_y \cap P_{y'} \neq \varnothing$, then $(\exists\alpha)$ $(\alpha \in I)$, $(\exists\alpha')$ $(\alpha' \in I)$ such that $G_\alpha \ni u = h_\alpha(y) = h_{\alpha'}(y') = u' \in G_{\alpha'}$; whence $G_\alpha \cap G_{\alpha'} \neq \varnothing$; but this relation is false; therefore,

$$P_y \cap P_{y'} = \varnothing .$$
$$y \neq y'$$

Moreover we have:

$$\bigcup_{y\in F} P_y = \bigcup_{y\in F} \Big(\bigcup_{\alpha\in I} \{h_\alpha(y)\}\Big) = \bigcup_{\alpha\in I} \Big(\bigcup_{y\in F} \{h_\alpha(\{y\})\}\Big) = \bigcup_{\alpha\in I} h_\alpha(F) = \bigcup_{\alpha\in I} G_\alpha = G_F .$$

Hence the family $(P_y)_{y\in F}$ is a partition of G_F.

Let R be the equivalence relation on G_F defined by this partition, i.e.,

$$\frac{G_F}{R} = \bigcup_{y\in F} P_y = \varinjlim F_\alpha .$$

On the other hand, we have

$$P_{y+y'} = \{(h_\alpha(y + y'))_{\alpha\in I}\} = (h_\alpha(y))_{\alpha\in I} + (h_\alpha(y'))_{\alpha\in I} = P_y + P_{y'} ;$$
$$P_{(y)} = -P_y ;$$
$$\{P_0 = (h_\alpha(0))_{\alpha\in I}\} \Rightarrow \{P_0 + P_y = P_y + P_0\}; \qquad P_y - P_y = P_0$$
$$P_y + P_{y'} = P_{y'} + P_y$$

and

$$(P_y + P_{y'}) + P_{y''} = P_y + (P_{y'} + P_{y''}) = P_y + P_{y'} + P_{y''} .$$

Therefore $\bigcup_{y\in F} P_y = G_F/R$ is an abelian group.

Let us now prove that the mapping $\theta : F \mapsto G_F/R$ defined by: $\theta(y) = P_y$ is an isomorphism of F onto $\varinjlim F_\alpha$.

We have

$$\theta(y + y') = \theta(y) + \theta(y'),$$

$$\theta(F) = \bigcup_{y \in F} P_y = \varinjlim F_\alpha = \frac{G_F}{R},$$

and

$$\ker \theta = \{0\}.$$

Therefore θ^{-1} is an isomorphism of $\varinjlim F_\alpha$ onto F, and we can identify F with $\varinjlim F_\alpha$ by means of this *canonical* isomorphism.

2. *The Direct Limit of Mappings*: $\hat{\nu} = \varinjlim \hat{\nu}_\alpha$

Let $(\mathfrak{M}_\alpha, \hat{g}_{\beta\alpha})$ be the direct system of σ-algebras as defined in Section 2, No. 1, and let $\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$ be the σ-algebra which is the direct limit of the family $(\mathfrak{M}_\alpha)$.

Let $(F_\alpha, f_{\beta\alpha})$ be a direct system of abelian groups and, let $\mathfrak{F} = \varinjlim F_\alpha$ be the abelian groups which is the direct limit of the family (F_α).

Let us, moreover, suppose that $F_\alpha = F$, $\forall \alpha \in I$, where F is a complete abelian group, and let us consider the direct system $(F_\alpha, i_{\beta\alpha})$, where $i_{\beta\alpha}$ is the identity mapping of F. Then (cf. No. 1) one can canonically identify F with $\varinjlim F_\alpha$. On the other hand, for each $\alpha \in I$, let $\hat{u}_\alpha$ be a mapping of $\mathfrak{M}_\alpha$ into F such that $\hat{u}_\alpha = \hat{u}_\beta \circ \hat{g}_{\beta\alpha}$. Then (cf. [2, Section 7, No. 6, Remark 2]) $(\hat{u}_\alpha)$ is a direct system of mappings defined by

$$\begin{tikzcd} \mathfrak{M}_\alpha \arrow[r, mapsto, "\hat{u}_\alpha"] \arrow[d, "\hat{g}_{\beta\alpha}"'] & F_\alpha = F \arrow[d, "i_{\beta\alpha}"] \\ \mathfrak{M}_\beta \arrow[r, mapsto, "\hat{u}_\beta"'] & F_\beta = F \end{tikzcd} \quad \Leftrightarrow i_{\beta\alpha} \circ \hat{u}_\alpha = \hat{u}_\beta \circ \hat{g}_{\beta\alpha},$$

FIGURE 6

and the mapping $\hat{u} : \mathfrak{M} \to F$, defined by $\hat{u}_\alpha = \hat{u} \circ \hat{\psi}_{\mathfrak{M}_\alpha}$, may be identified with the direct limit of the direct system of mappings $(\hat{u}_\alpha)_{\alpha \in I}$. Under these conditions one can write $\hat{u} = \varinjlim \hat{u}_\alpha$. In particular, let us suppose that, for each $\alpha \in I$, $\lambda_\alpha = \hat{u}_\alpha$ is a countable additive mapping of $\mathfrak{M}_\alpha$ into F. Then $\lambda = \varinjlim \lambda_\alpha$ is such that $\lambda_\alpha = \lambda \circ \hat{\psi}_{\mathfrak{M}_\alpha}$, and λ is an additive mapping (cf. Theorem 1, Section 3, No. 1) of $\mathfrak{M}$ into F.

3. *The Countable Additive Mapping* $\lambda = \varinjlim \lambda_\alpha$

Let us now prove that λ is a countable additive mapping of $\mathfrak{M}$ into F; that is, for each sequence $(X_\nu)_{\nu \in \mathbf{N}}$ of distinct elements of $\mathfrak{M}$, we have

$$\lambda\left(\sum_{\nu \in \mathbf{N}} X_\nu\right) = \sum_{\nu \in \mathbf{N}} \lambda(X_\nu).$$

Indeed,

$$\{X_\nu \in \mathfrak{M}\} \Rightarrow \{\exists \alpha_\nu \in I \leadsto X_\nu = \hat{\psi}_{\mathfrak{M}_{\alpha_\nu}}(X_{\alpha_\nu})\}$$

$$\Rightarrow \Big\{\bigcup_{\nu\in\mathbf{N}} X_\nu = \bigcup_{\nu\in\mathbf{N}} \hat{\psi}_{\mathfrak{M}_{\alpha_\nu}}(X_{\alpha_\nu}) = \bigcup_{\nu\in\mathbf{N}} \hat{\psi}(X_{\alpha_\nu}) = \bigcup_{\nu\in\mathbf{N}} \hat{\psi}\langle\{X_{\alpha_\nu}\}\rangle$$

$$= \hat{\psi}\left\langle \bigcup_{\nu\in\mathbf{N}} \{X_{\alpha_\nu}\}\right\rangle = \hat{\psi}\left\langle \Big\{\bigcup_{\nu\in\mathbf{N}} X_{\alpha_\nu}\Big\}\right\rangle = \hat{\psi}\left(\bigcup_{\nu\in\mathbf{N}} X_{\alpha_\nu}\right)\Big\}.$$

On the other hand, $\{X_\nu \in \mathfrak{M}\} \Rightarrow \{\bigcup_{\nu\in\mathbf{N}} X_\nu \in \mathfrak{M}\}$, since $\mathfrak{M}$ is a σ-algebra (cf. Theorem 1 of Section 3, No. 1), therefore, there exists (cf. [6, Section 7, No. 5, Lemma 1]) a unique $\alpha \in I$ such that $\bigcup_{\nu\in\mathbf{N}} X_\nu = \hat{\psi}(X_\alpha)$, where $X_\alpha \in \mathfrak{M}_\alpha$; whence

$$\Big\{\hat{\psi}\left(\bigcup_{\nu\in\mathbf{N}} X_{\alpha_\nu}\right) = \hat{\psi}(X_\alpha) \Leftrightarrow \hat{\phi}\left(\bigcup_{\nu\in\mathbf{N}} X_{\alpha_\nu}\right) = \hat{\phi}(X_\alpha)$$

by definition of $\hat{R}$ (cf. Section 1, No. 2)$\} \Leftrightarrow \{\phi\langle\bigcup_{\nu\in\mathbf{N}} X_{\alpha_\nu}\rangle = \phi\langle X_\alpha\rangle\} \Leftrightarrow \{\exists X_\alpha^{(\nu)} \subset X_\alpha$ for each $\nu \in \mathbf{N} \leadsto \phi\langle X_{\alpha_\nu}\rangle = \phi\langle X_\alpha^{(\nu)}\rangle\}$.

Whence

$$\bigcup_{\nu\in\mathbf{N}} X_\nu = \hat{\psi}\left(\bigcup_{\nu\in\mathbf{N}} X_{\alpha_\nu}\right) = \hat{g}\left(\hat{\phi}\left(\bigcup_{\nu\in\mathbf{N}} X_{\alpha_\nu}\right)\right) = \hat{g}\left(\phi\left\langle\bigcup_{\nu\in\mathbf{N}} X_{\alpha_\nu}\right\rangle\right) = \hat{g}\left(\bigcup_{\nu\in\mathbf{N}} \phi\langle X_{\alpha_\nu}\rangle\right)$$

$$= \hat{g}\left(\bigcup_{\nu\in\mathbf{N}} \phi\langle X_\alpha(\nu)\rangle\right) = \hat{g}\left(\phi\left\langle\bigcup_{\nu\in\mathbf{N}} X_\alpha^{(\nu)}\right\rangle\right) = \hat{g}\left(\hat{\phi}\left(\bigcup_{\nu\in\mathbf{N}} X_\alpha^{(\nu)}\right)\right) = \hat{\psi}\left(\bigcup_{\nu\in\mathbf{N}} X_\alpha^{(\nu)}\right).$$

Therefore,

$$\lambda\left(\bigcup_{\nu\in\mathbf{N}} X_\nu\right) = \lambda\left(\hat{\psi}_{\mathfrak{M}_\alpha}\left(\bigcup_{\nu\in\mathbf{N}} X_\alpha^{(\nu)}\right)\right) = \lambda_\alpha\left(\bigcup_{\nu\in\mathbf{N}} X_\alpha^{(\nu)}\right).$$

But

$$\{\underset{\nu\neq\mu}{X_\nu \neq X_\mu}\} \Rightarrow \{\underset{\nu\neq\mu}{X_\nu \cap X_\mu} = \phi \text{ in } M\} \Rightarrow \{\underset{\nu\neq\mu}{X_{\alpha_\nu} \cap X_{\alpha_\mu}} = \phi\} \Rightarrow \{X_\alpha^{(\nu)} \cap X_\alpha^{(\mu)} = \phi\}$$

$$\Rightarrow \Big\{\lambda_\alpha\left(\bigcup_{\nu\in\mathbf{N}} X_\alpha^{(\nu)}\right) = \sum_{\nu\in\mathbf{N}} \lambda_\alpha(X_\alpha^{(\nu)})\Big\}$$

$$\Rightarrow \Big\{\lambda\left(\bigcup_{\nu\in\mathbf{N}} X_\nu\right) = \sum_{\nu\in\mathbf{N}} \lambda_\alpha(X_\alpha^{(\nu)})) = \sum_{\nu\in\mathbf{N}} \lambda(\hat{\psi}_{\mathfrak{M}_\alpha}(X_\alpha^{(\nu)})$$

$$= \sum_{\nu\in\mathbf{N}} \lambda(\hat{\psi}_{\mathfrak{M}_{\alpha_\nu}}(X_{\alpha_\nu})) = \sum_{\nu\in\mathbf{N}} \lambda(X_\nu)\Big\}.$$

$$\Rightarrow \{\lambda \text{ is a countable additive mapping of } \mathfrak{M} \text{ into } F\}.$$

We have also proved

THEOREM 2. *Under the hypothesis of Theorem* 1, *Section* 2, *and Theorem* 1, *Section* 3, *let* $(\lambda_\alpha)_{\alpha\in I}$ *be a direct system of measures relative to* I, *with value in a complete abelian group* F. *More precisely, let* $\lambda_\alpha : \mathfrak{M}_\alpha \mapsto F$, *be a measure on* $\mathfrak{M}_\alpha$, *with values in* F, *for each* $\alpha \in I$. *Let* $\lambda = \varinjlim \lambda_\alpha$ *be the direct limit of the direct system of measures* $(\lambda_\alpha)_{\alpha\in I}$. *Under these conditions;* λ *is a measure on* $E = G/R$, *with values in* F. *Then* λ *is called the direct limit measure of the direct system of measures* $(\lambda_\alpha)_{\alpha\in I}$.

Therefore, we have

THEOREM 3. $\{(E_\alpha, \mathfrak{M}_\alpha, \lambda_\alpha) =$ *direct system of measure spaces*$\} \Rightarrow \{(E; \mathfrak{M}, \lambda) = (\varinjlim E_\alpha, \varinjlim \mathfrak{M}_\alpha, \varinjlim \lambda\alpha)$ *is a measure space*$\}$.

We shall say that $(E, \mathfrak{M}, \lambda)$ is the measure space which is the direct limit of the family $(E_\alpha, \mathfrak{M}_\alpha, \lambda_\alpha)$ of measure spaces.

In particular, we have

THEOREM 4.

$$\{(E_\alpha, \mathfrak{M}_\alpha, p_\alpha)_{\alpha\in I} = \textit{family of probability spaces}\}$$
$$\Rightarrow \{(\varinjlim E_\alpha, \varinjlim \mathfrak{M}_\alpha, \varinjlim p_\alpha) = \textit{probability space}\}.$$

A further communication will be devoted to the study and applications of direct limits of probability spaces.

REFERENCES

[1] RAO, M. M. (1971). Projective limits of probability spaces. *J. Multivariate Anal.* **1** 28–57.

[2] CHOKSI, J. R. (1958). Inverse limits of measure spaces. *Proc. Lond. Math. Soc.* **8** 321–342.

[3] SCHEFFER, C. L. (1969). Limits of directed projective systems of probability spaces. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **13** 60–80.

[4] BOURBAKI, N. (1966). *Théorie des Ensembles.* Chaps. I, II. Hermann, Paris.

[5] VASILACH, S. (1971). On the extention of the equivalence relations to the sets of subsets. *Atti Della Academia Nazionale dei Lincei,* Serie VIII, Vol. L, Fasc. 2.

[6] BOURBAKI, N. (1963). *Théorie des Ensembles.* Chap. III. Hermann, Paris, 1963.